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Comments on defects in the a_r Toda field theories

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Abstract
A simple, basic argument is given, based solely on energy–momentum considerations, to recover conditions under which a_r affine or conformal Toda field theories can support defects of integrable type. Associated triangle relations are solved to provide expressions for transmission matrices that generalize previously known examples calculated for the sine-Gordon model and the a_2 affine Toda model.

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1. Introduction

The study of defects, or impurities, within integrable field theory was initiated nearly 15 years ago by Delfino *et al* [1]. They pointed out that, with some natural assumptions, it would not be possible for an integrable system to encompass a defect, such as a delta-function impurity, allowing both reflection and transmission compatible with a non-trivial bulk scattering matrix [2]. One may question the assumptions (see, for example, [3]), or analyse those types of defects that are compatible with the bulk S-matrix, for example, those that are purely transmitting (within the sine-Gordon model this began with some work of Konik and LeClair [4]). It is already known from numerical studies of phenomena in the classical sine-Gordon model that a delta-function impurity is unlikely to be integrable (see, for example, [5]), but it was pointed out in [6] that another possibility was to allow field discontinuities. At first sight, this appears to be quite drastic and unlikely to lead anywhere. However, it turned out that discontinuities could be permitted provided the fields on either side of the discontinuity were 'sewn' together appropriately. Moreover, the sewing conditions that guaranteed integrability were closely related to Bäcklund transformations, a fact that emerged not only in the sine-Gordon model but also for the subset of affine Toda models defined in terms of the root data of a_r (the sine-Gordon model itself corresponding to a_1) [7]. For the sine-Gordon model itself it was possible to analyse in [8] the relationship between the classical and quantum theories possessing this type of discontinuity—which are really more akin to 'shocks', and sometimes referred to as 'jump-defects' to distinguish them from delta-function impurities—thereby providing a framework for the Konik–LeClair transmission matrix and various means of checking it, including perturbative calculations of the transmission factors for breathers [9].

One purpose of this paper is to provide simple and reasonably general arguments leading to the sewing conditions previously proposed for the affine Toda field theories. It appears that the a_r models are special and we have not yet found a way to generalize the argument to all the other models, or indeed to find an alternative. This is a slightly frustrating situation, perhaps indicating simply a lack of imagination, because in other contexts members of the whole class of affine Toda models, apart from relatively small details depending on the choice of the root system, have very similar properties. In passing, it is remarked how in the context of the conformal Toda models a sequence of defects can transform one model into another. The illustrative example of this behaviour shows how an a_r model can be reduced to an a_{r-1} model together with a free massless field. Once this is shown to be integrable (and an argument is provided in section 3), combinations of defects can be used to construct mixtures of conformal models. The simplest example of this is the well-known relationship between the Liouville model and a massless free field.

A second purpose is to make progress towards completing the story that was begun in [10]. There, besides general remarks that applied to each member of the a_r class of affine Toda models, it proved possible to solve in detail the triangle relations for a_2 affine Toda theory and to describe some of the intriguing properties of the transmission matrix, especially those surrounding the curiously different character of the interactions between the defect and the two types of soliton (perhaps better regarded as soliton and anti-soliton). Here, the techniques are generalized to calculate transmission matrices for the a_r affine Toda models and to investigate some of their properties, particularly with regard to unstable bound states.

2. Energy and momentum revisited

In the bulk, $-\infty < x < \infty$, an affine Toda field theory corresponding to the root data of a Lie algebra g is described by the Lagrangian density

$$\mathcal{L}_u = \frac{1}{2}(\partial_\mu u \cdot \partial^\mu u) - \frac{m^2}{\beta^2}\sum_{j=0}^{r} n_j (\mathrm{e}^{\beta\alpha_j\cdot u} - 1), \tag{2.1}$$

where m and β are constants and r is the rank of the algebra. The set of vectors $\{\alpha_j\}$ with $j = 1, \ldots, r$ are the simple roots of g, while α_0 is an extra root, defined by $\alpha_0 = -\sum_{j=1}^{r} n_j\alpha_j$. The integers $\{n_j\}$ are a set of integers characteristic of each affine Toda model. Each set of roots is associated with an affine Dynkin–Kač diagram, which encodes the inner products among the simple roots $\{\alpha_j\}$ including the extra root α_0. Finally, the field $u = (u_1, u_2, \ldots, u_r)$ takes values in the r-dimensional Euclidean space spanned by the simple roots $\{\alpha_j\}$. The affine Toda models are massive and integrable. However, if the term in the Lagrangian with $j = 0$ is omitted, then the theory described by the density Lagrangian (2.1) is conformal and called a conformal Toda field theory. All these models possess a Lax pair representation and they have been extensively investigated in the past, both classically and in the quantum domain. For further details concerning the affine Toda field theories, see [11, 12], and the review [13]; for further details on the conformal Toda models see, for instance, [14, 15], and references therein.

In this paper, a Lagrangian density of the following type:

$$\mathcal{L}_D = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v - \delta(x)\mathcal{W}, \tag{2.2}$$

which couples together two sets of r scalar fields u, v by means of a defect located in $x = 0$, will be investigated.

The purpose of this section is to start from first principles to determine for which Toda field theories there is a set of defect conditions that will allow exchange of energy–momentum between a defect and the fields on either side of it. The result is a little surprising.

Consider the standard expressions for the energy and momentum carried by the fields u and v:

$$\mathcal{E} = \int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}(u_x \cdot u_x) + \frac{1}{2}(u_t \cdot u_t) + U(u)\right) + \int_{0}^{\infty} \mathrm{d}x \left(\frac{1}{2}(v_x \cdot v_x) + \frac{1}{2}(v_t \cdot v_t) + V(v)\right)$$

and

$$\mathcal{P} = \int_{-\infty}^{0} \mathrm{d}x (u_x \cdot u_t) + \int_{0}^{\infty} \mathrm{d}x (v_x \cdot v_t),$$

where, for the time being, the potentials for the fields u and v remain unspecified. Differentiating with respect to time, using the equations of motion for the two fields in their respective domains, and assuming no contributions from $x = \pm\infty$ give, one has, respectively,

$$\dot{\mathcal{E}} = u_x \cdot u_t|_{x=0} - v_x \cdot v_t|_{x=0} \tag{2.3}$$

and

$$\dot{\mathcal{P}} = \left(\tfrac{1}{2}(u_x \cdot u_x) + \tfrac{1}{2}(u_t \cdot u_t) - U(u)\right)_{x=0} - \left(\tfrac{1}{2}(v_x \cdot v_x) + \tfrac{1}{2}(v_t \cdot v_t) - V(v)\right)_{x=0}. \tag{2.4}$$

Energy–momentum will be exchangeable with the defect provided each of these may be expressed as time derivatives of functions of the fields evaluated at $x = 0$. Consider first (2.3) and suppose, at $x = 0$, the rather general condition relating space derivatives,

$$u_x = Au_t + Bv_t + X(u, v), \qquad v_x = Cu_t + Dv_t + Y(u, v), \tag{2.5}$$

where A, B, C, D are matrices and X, Y are vector functions of u and v. Then,

$$\dot{\mathcal{E}} = u_t \cdot Au_t + u_t \cdot Bv_t - v_t \cdot Cu_t - v_t \cdot Dv_t + u_t \cdot X - v_t \cdot Y,$$

and this will be a total time derivative provided

$$C = B^T, \qquad A = -A^T, \qquad D = -D^T, \qquad X = -\nabla_u \mathcal{D}, \qquad Y = \nabla_v \mathcal{D}, \tag{2.6}$$

where $\mathcal{D}$ is also a function of u and v evaluated at $x = 0$. Under these circumstances, $\mathcal{E} + \mathcal{D}$ is conserved.

Next, consider (2.4). A similar computation places further constraints, namely

$$1 - A^2 = BB^T, \qquad 1 - D^2 = B^T B, \qquad AB + BD = 0 \tag{2.7}$$

and

$$(\nabla_u \mathcal{D} \cdot \nabla_u \mathcal{D}) - (\nabla_v \mathcal{D} \cdot \nabla_v \mathcal{D}) = 2(U - V), \tag{2.8}$$

together with the requirement

$$(A\nabla_u \mathcal{D} - B\nabla_v \mathcal{D}) = \nabla_u \Omega, \qquad (B^T \nabla_u \mathcal{D} - D\nabla_v \mathcal{D}) = -\nabla_v \Omega, \tag{2.9}$$

where Ω is another function of the fields u and v evaluated at $x = 0$. Under these circumstances, $\mathcal{P} + \Omega$ is conserved. Provided all these constraints may be simultaneously satisfied energy and momentum will be conserved once specific contributions coming from the defect itself are taken into account.

The first expression in (2.7) may be rewritten as follows:

$$(1 + A^T)(1 + A) = BB^T. \tag{2.10}$$

Since A is real and antisymmetric, its eigenvalues are purely imaginary and hence $(1 - A)$ is invertible. Then

$$1 = (1 - A)^{-1} B((1 - A)^{-1} B)^T, \tag{2.11}$$

which implies that $(1 - A)^{-1} B$ is orthogonal and $B = (1 - A)O$, with $O \in O(r)$.

At this stage, if it is further assumed that the various matrices are independent of u and v, it can be remarked also that the set of boundary conditions of the type (2.5) follows from the Lagrangian density (2.2) with the choice

$$\mathcal{W} = \left(\tfrac{1}{2}u_t \cdot Au - \tfrac{1}{2}v_t \cdot Dv + u_t \cdot Bv + \mathcal{D}(u, v)\right). \tag{2.12}$$

Making an orthogonal transformation on the field v, this expression may be rewritten as

$$\mathcal{W} = \left(\tfrac{1}{2}u_t \cdot Au - \tfrac{1}{2}v_t \cdot D'v + u_t \cdot (1-A)v + \mathcal{D}(u, v)\right). \tag{2.13}$$

As a consequence, the third expression in (2.7) reads

$$A(1-A) + (1-A)D' = 0, \tag{2.14}$$

which implies $D' = -A$ and

$$B = (1-A), \qquad B^T = (1+A) = (2-B). \tag{2.15}$$

It is easy to verify that the second equation in (2.7) is automatically satisfied. It is worth pointing out that an orthogonal transformation on the field v translates simply into a change of base for the simple roots appearing in the expression of the potential $V(v)$. Note that a similar result would have been obtained by starting from the second equation in (2.7) and applying an orthogonal transformation to the field u.

Results (2.14) and (2.15) allow constraints (2.9) to be rewritten in terms of the matrix A alone. Then, by computing all the second derivatives of Ω and requiring consistency, the following additional constraints are obtained:

$$\begin{aligned}
&A_{lj}\frac{\partial^2\mathcal{D}}{\partial u_k \partial u_j} - (1-A)_{lj}\frac{\partial\mathcal{D}}{\partial u_k \partial v_j} = A_{kj}\frac{\partial^2\mathcal{D}}{\partial u_l \partial u_j} - (1-A)_{kj}\frac{\partial\mathcal{D}}{\partial u_l \partial v_j},\\
&(1+A)_{lj}\frac{\partial\mathcal{D}}{\partial v_k \partial u_j} + A_{lj}\frac{\partial^2\mathcal{D}}{\partial v_k \partial v_j} = (1+A)_{kj}\frac{\partial\mathcal{D}}{\partial v_l \partial u_j} + A_{kj}\frac{\partial^2\mathcal{D}}{\partial v_l \partial v_j},\\
&-A_{lj}\frac{\partial^2\mathcal{D}}{\partial v_k \partial u_j} + (1-A)_{lj}\frac{\partial\mathcal{D}}{\partial v_k \partial v_j} = (1+A)_{kj}\frac{\partial\mathcal{D}}{\partial u_l \partial u_j} + A_{kj}\frac{\partial^2\mathcal{D}}{\partial u_l \partial v_j}.
\end{aligned} \tag{2.16}$$

Since u, v are Toda-like fields, solutions for the defect potential $\mathcal{D}$ should have the form

$$\exp(a \cdot u + b \cdot v),$$

where a, b are vectors needing to be specified. Using this fact the constraints (2.16) reduce to the following tensorial expressions:

$$a \otimes (-Aa + (1-A)b) = (-Aa + (1-A)b) \otimes a \tag{2.17}$$

$$((1+A)a + Ab) \otimes b = b \otimes ((1+A)a + Ab) \tag{2.18}$$

$$b \otimes (-Aa + (1-A)b) = ((1+A)a + Ab) \otimes a. \tag{2.19}$$

The first two relations imply $(-Aa + (1-A)b) = \alpha a$, and $((1+A)a + Ab) = \beta b$, respectively, with α, β constants. Hence (2.19) forces $\alpha = \beta$. Constraints for the vectors a and b are provided by (2.17) and (2.18), since they may be rewritten as follows:

$$a = (1+A)^{-1}(\alpha - A)b, \qquad (1-\alpha^2)b = 0. \tag{2.20}$$

Clearly $\alpha^2 = 1$ since the possibility $b = 0$ is uninteresting since it also implies $a = 0$ and a trivial $\mathcal{D}$. Choosing $\alpha = 1$ and setting $a = (1-A)x/2, b = (1+A)x/2$ the defect potential $\mathcal{D}$ has the form $\exp((u \cdot (1-A) + v \cdot (1+A))x/2)$, while for $\alpha = -1$, setting $a = -b = y/2$, the defect potential $\mathcal{D}$ has the form $\exp((u-v) \cdot y/2)$. The vectors x, y are not yet determined.

Information collected so far implies a general expression for the defect potential, namely,

$$\mathcal{D} = \sum_k p_k \, \mathrm{e}^{(u-v)\cdot y_k/2} + \sum_l q_l \, \mathrm{e}^{(u\cdot(1-A)+v\cdot(1+A))x_l/2}, \tag{2.21}$$

where p_k, q_l are constant coefficients.

This expression can now be used to investigate the last constraint (2.8), which links the defect potential to the bulk potentials for the fields u and v. After some algebra the constraint turns out to be

$$\sum_{k,l} p_k q_l (y_k \cdot x_l) \, \mathrm{e}^{u\cdot(y_k+(1-A)x_l)/2+v\cdot(-y_k+(1+A)x_l)/2} = 2(U(u) - V(v)). \tag{2.22}$$

Before analysing this expression, note that on the left-hand side of (2.22) there can be no repeated exponents. In fact, if two exponents were to be the same, given two pairs of vectors (x_i, y_i) and (x_j, y_j), the following condition would hold

$$y_i + (1-A)x_i = y_j + (1-A)x_j, \qquad -y_i + (1+A)x_i = -y_j + (1+A)x_j,$$

implying $x_i = x_j$ and, therefore, $y_i = y_j$. Hence, to each different pair of vectors (x_i, y_i) and (x_j, y_j) there correspond two different exponents on the left-hand side of expression (2.22). Note also that, in principle, the two bulk potentials $U(u)$ and $V(v)$ could belong to two different Toda-like models, provided the number r of fields either side of the defect is the same.

Denote by $\{\alpha_i\}$, $\{\alpha_i'\}$ the two sets of simple roots of the Lie algebras associated with the models on the left and on the right of the defect respectively, together with—if required—the extended root. Since the left-hand side of (2.22) must be equal to the difference of two Toda-like bulk potentials, there must exist four sets of vectors $\{x_l\}$, $\{x_l'\}$, $\{y_k\}$, $\{y_k'\}$ such that

$$y_i = (1+A)x_i, \qquad (y_i \cdot x_i) \neq 0, \qquad y_i \in \{y_k\}, \qquad x_i \in \{x_l\} \quad \text{to give} \quad \mathrm{e}^{u\cdot x_i} \tag{2.23}$$

and

$$y_i' = -(1-A)x_i', \qquad (y_i' \cdot x_i') \neq 0, \qquad y_i' \in \{y_k'\}, \qquad x_i' \in \{x_l'\} \quad \text{to give} \quad \mathrm{e}^{v\cdot x_i'}. \tag{2.24}$$

Clearly, $x_i \equiv \alpha_i$, $x_i' \equiv \alpha_i'$. The exponential terms obtained in (2.23) and (2.24) correspond to pieces necessary for building the two bulk potentials $U(u)$ and $V(v)$. Obviously, they are the only possibilities allowed in (2.22). This means that any other terms that might arise in (2.22) because of particular choices of vectors $\{\alpha_l\}$, $\{\alpha_l'\}$, $\{y_k\}$, $\{y_k'\}$ must have coefficients equal to zero. Note, the possibility of having two exponentials that cancel is ruled out by the fact that any given exponential may only appear once on the left-hand side of (2.22), as established above.

To analyse further expression (2.22), consider first the case in which the two sets $\{\alpha_l\}$ and $\{\alpha_l'\}$ coincide. For a given $y_i \in \{y_k\}$, it could happen that

$$(y_i \cdot \alpha_j) = 0 \qquad \forall \alpha_j \in \{\alpha_l\} \, j \neq i. \tag{2.25}$$

Hence, it would be possible to write $y_i = (\alpha_i \cdot \alpha_i) w_i$, where w_i is a fundamental highest weight of the Lie algebra associated with the Toda-like model on both sides of the defect[1]. Note that this choice satisfies condition (2.25) and, because of (2.23), implies $\alpha_j \cdot A\alpha_i = -\alpha_j \cdot \alpha_i$, $\forall j \neq i$.

As an alternative to condition (2.25), suppose that for $y_i \in \{y_k\}$ there exists at least one $\alpha_j \in \{\alpha_l\}$ such that $(y_i \cdot \alpha_j) \neq 0$ with $j \neq i$. In the most general case, the exponent associated with the pair (y_i, α_j) in (2.22) will be a combination of both fields u and v. However, such a term is not allowed. A way out is to suppose, in addition, $y_i = -(1-A)\alpha_j$. Then, $y_i \equiv y_j'$

[1] In the present case they coincide since they have the same set of simple roots.

with $y'_j \in \{y'_k\}$, therefore the resulting exponential is permitted since it coincides with a term of the bulk potential $V(v)$. Then,

$$y_i = 2\alpha_i - (1-A)\alpha_i, \qquad y_i = -2\alpha_j + (1+A)\alpha_j. \tag{2.26}$$

Multiplying these two expressions by α_i and α_j, respectively, leads to

$$(\alpha_i \cdot \alpha_i) = (\alpha_j \cdot \alpha_j) = (y_i \cdot \alpha_i) = -(y_i \cdot \alpha_j). \tag{2.27}$$

Hence it is possible to write $y_i = (\alpha_i \cdot \alpha_i)(w_i - w_j)$ where w_i, w_j are fundamental highest weights of the Lie algebra associated with the two Toda-like models. Note, this situation can only occur when the roots α_i, α_j have the same length.

In summary, expression (2.22) is solved by choosing three sets $\{x_l\}$, $\{y_k\}$ and $\{y'_k\}$ such that

$$\{x_l\} \equiv \{\alpha_l\},$$

where $\{\alpha_l\}$ is a set of simple roots together with—if included—the extended root, and $y_i \in \{y_k\}$, $y'_i \in \{y'_k\}$ can have one of the following forms ((a), (c) for y_i and (b), (c) for y'_i, respectively):

$$(a) \quad y_i = (1+A)\alpha_i = (\alpha_i \cdot \alpha_i)w_i, \qquad (\alpha_j \cdot A\alpha_i) = -(\alpha_j \cdot \alpha_i) \qquad j \neq i \tag{2.28}$$

$$(b) \quad y'_i = -(1-A)\alpha_i = -(\alpha_i \cdot \alpha_i)w_i, \qquad (\alpha_j \cdot A\alpha_i) = (\alpha_j \cdot \alpha_i) \qquad j \neq i \tag{2.29}$$

$$\begin{aligned}(c) \quad & y_i \equiv y'_j = (1+A)\alpha_i = -(1-A)\alpha_j = (\alpha_i \cdot \alpha_i)(w_i - w_j),\\ & (\alpha_j \cdot A\alpha_i) = -(\alpha_i \cdot \alpha_i) - (\alpha_j \cdot \alpha_i), \qquad (\alpha_i \cdot \alpha_i) \equiv (\alpha_j \cdot \alpha_j) \qquad j \neq i.\end{aligned} \tag{2.30}$$

Clearly, the possibility (c) (2.30) implies an overlapping among the elements in the two sets $\{y_k\}$ and $\{y'_k\}$. To decide which form among the possibilities listed above to choose for each vector in these two sets, it is worth noting that because of (2.23) and (2.24), the following constraint holds

$$y_i - y'_i = 2\alpha_i. \tag{2.31}$$

To satisfy this constraint, the only possible combinations for the explicit forms of the pair (y_i, y'_i) are

$$((a),(b)) \quad y_i - y'_i = 2\alpha_i = 2(\alpha_i \cdot \alpha_i)w_i \tag{2.32}$$

$$((a),(c)) \quad y_i - y'_i = 2\alpha_i = (\alpha_i \cdot \alpha_i)(2w_i - w_j), \qquad (1+A)\alpha_j = -(1-A)\alpha_i, \quad i \neq j \tag{2.33}$$

$$((c),(b)) \quad y_i - y'_i = 2\alpha_i = (\alpha_i \cdot \alpha_i)(2w_i - w_m), \qquad (1+A)\alpha_i = -(1-A)\alpha_m, \quad i \neq m \tag{2.34}$$

$$\begin{aligned}((c),(c)) \quad & y_i - y'_i = 2\alpha_i = (\alpha_i \cdot \alpha_i)(2w_i - w_j - w_m),\\ & (1+A)\alpha_i = -(1-A)\alpha_j, \qquad (1+A)\alpha_m = -(1-A)\alpha_i, \qquad i \neq j \neq m.\end{aligned} \tag{2.35}$$

Clearly the combination (2.32) can only appear if the root is unconnected to all the others since there is no simple root that coincides up to scaling with a single fundamental highest weight except for a_1. Such cases will not be considered further here since the Dynkin diagram would have at least one disconnected spot. In addition, by looking at the other combinations, it is clear that each node of the Dynkin diagram associated with the set of roots $\{\alpha_l\}$ must have no more than two linked neighbours since $\alpha_i = (2w_i - w_m)$ or $\alpha_i = (2w_i - w_j - w_m)$ at most[2]. This fact, together with the observation that the possibility (c) (2.30) might only happen when the roots involved have the same length, implies that the only Toda-like field theories allowed are those associated with Lie algebras of type a_r. This is a surprising result and the only assumption made to simplify the discussion was to suppose the matrices A, B, C, D were independent of u and v. Relaxing this considerably complicates the discussion yet might be necessary to be able to apply the same type of arguments to Toda models based on other root systems.

From now on consider only the Lie algebra a_r. The simple roots together with the extended (lowest) root can be written in term of the fundamental highest weights $w_i i = 1, \dots, r$ via

$$\alpha_i = (2w_i - w_{i+1} - w_{i-1}) \qquad i = 1, \dots, r, \qquad w_0 \equiv w_{r+1} = 0. \tag{2.36}$$

Consider two affine a_r Toda theories on either side of the defect, then a total momentum is conserved provided $\{\alpha_l\}$ is the set of simple roots of a_r together with the extended root, and the elements of the sets $\{y_k\}, \{y'_k\}$ with $k, k' = 0, \dots, r$ are as follows:

$$y_i = y'_{i-1} \qquad i = 2, \dots, r, \qquad y_1 = y'_0, \qquad y_0 = y'_r. \tag{2.37}$$

It can be noted that the pairs $(y_1, y'_1), (y_r, y'_r)$ correspond to the combinations (2.33) and (2.34), respectively, while all other pairs correspond to the case (2.35). The matrix B may be written explicitly in terms of the fundamental weights of the algebra a_r

$$B = (1 - A) = 2 \sum_{a=1}^{r} (w_a - w_{a+1}) w_a^T. \tag{2.38}$$

A formula first obtained by other means in [7].

Note that the matrix B can be replaced by its transpose to provide another solution, which is represented by

$$B = (1 - A) = 2 \sum_{a=1}^{r} w_a (w_a - w_{a+1})^T. \tag{2.39}$$

Then

$$y_i = y'_{i+1} \qquad i = 1, \dots, r, \qquad y_0 = y'_1, \tag{2.40}$$

and the pairs $(y_1, y'_1), (y_r, y'_r)$ correspond to the combinations (2.34) and (2.33), respectively, while all other pairs correspond to the case (2.35).

Setting $p_i = \sigma$ for all $p_i \in \{p_k\}$ and $q_i = 1/\sigma$ for all $q_i \in \{q_l\}$, the defect potential (2.21) reads

$$\mathcal{D} = \sum_{k=0}^{r} \frac{1}{\sigma} \, \mathrm{e}^{(u-v)\cdot(1+A)\alpha_k/2} + \sum_{l=0}^{r} \sigma \, \mathrm{e}^{(u\cdot(1-A)+v\cdot(1+A))\alpha_l/2}. \tag{2.41}$$

Next, consider instead two a_r conformal Toda theories on either side of the defect. A solution to expression (2.22) is represented, for example, by the matrix B as in (2.38) with $\{\alpha_l\}$ the set of simple roots of a_r and the two sets $\{y_k\}, \{y'_k\}$ as in (2.37) with $k, k' = 1, \dots, r$. In other words, the vectors y_0 and y'_0 have been omitted. The defect potential is

$$\mathcal{D} = \sum_{k=1}^{r} \frac{1}{\sigma} \, \mathrm{e}^{(u-v)\cdot(1+A)\alpha_k/2} + \frac{1}{\sigma} \, \mathrm{e}^{-(u-v)\cdot(1-A)\alpha_r/2} + \sum_{l=1}^{r} \sigma \, \mathrm{e}^{(u\cdot(1-A)+v\cdot(1+A))\alpha_l/2}. \tag{2.42}$$

[2] Setting $(\alpha_i \cdot \alpha_i) = 2$.

But, from (2.38), it is easy to realize that $-(1-A)\alpha_r = (1+A)\alpha_0$, where α_0 is the extended root, hence

$$\mathcal{D} = \sum_{k=0}^{r} \frac{1}{\sigma}\, \mathrm{e}^{(u-v)\cdot(1+A)\alpha_k/2} + \sum_{l=1}^{r} \sigma\, \mathrm{e}^{(u\cdot(1-A)+v\cdot(1+A))\alpha_l/2}. \tag{2.43}$$

Finally, there is another intriguing possibility. Consider, for example, the matrix (2.38). Then a solution to expression (2.22) is provided by a set $\{\alpha_l\}$ of simple roots of the Lie algebra a_r and two sets $\{y_k\}k = 1, \ldots, r$ and $\{y'_k\}$ with $k' = 1, \ldots, (r-1)$, whose elements satisfy (2.37). This time the vectors y_0, y'_0 and y'_r are missing. Under these circumstances, the defect potential reads

$$\mathcal{D} = \sum_{k=1}^{r} \frac{1}{\sigma}\, \mathrm{e}^{(u-v)\cdot(1+A)\alpha_k/2} + \sum_{l=1}^{r} \sigma\, \mathrm{e}^{(u\cdot(1-A)+v\cdot(1+A))\alpha_l/2}. \tag{2.44}$$

This situation allows the conservation of momentum for a defect system with an a_r conformal Toda field theory on the left of the defect and an a_{r-1} conformal Toda theory plus a free massless field, on the right. This case is allowed since all the fields involved are massless. In fact, it is possible to think of the algebra a_{r-1} embedded within the a_r algebra and the defect peels off the simple root at one end of the Dynkin diagram. If there was a sequence of r defects it would be possible to reduce the a_r conformal Toda theory to a collection of free massless fields, a situation that was not noted before.

To clarify this point and verify that the conservation of a total momentum implies integrability, the Lax pair construction for this specific case will be explored in the following section.

Before concluding this section, it is worth adding a few words on the possibility that the two sets $\{\alpha_l\}$ and $\{\alpha'_l\}$ do not in fact coincide. First of all, it is clear that these two sets cannot be completely disjoint. In fact, if this were the case, then, in addition to (2.23) and (2.24), for each $y_i \in \{y_k\}$ it would also be required that

$$(y_i \cdot \alpha'_i) = 0, \qquad \forall \alpha'_i \in \{\alpha'_k\}. \tag{2.45}$$

However, the simple roots in $\{\alpha'_k\}$ are r linearly independent vectors in an r-dimensional space, hence condition (2.45) could be satisfied only provided $y_i = 0$, which is false.

Actually, even a partial identification among the elements of the sets $\{\alpha_l\}$ and $\{\alpha'_l\}$ is not possible. To see this, consider two non-orthogonal simple roots α and β, $(\alpha\cdot\beta \neq 0)$, such that $y = (1+A)\cdot\alpha$ and $z = (1+A)\cdot\beta$. In this way they will realize two exponents of the type (2.23), which are part of the potential $U(u)$. Since the roots are not orthogonal and the matrix A is antisymmetric, it follows the scalar products $(\beta\cdot y)$ and $(\alpha\cdot z)$ cannot both be zero. Suppose $(\beta\cdot y)\neq 0$. This means that given the vector y there are two simple roots α and β whose scalar product with y differs from zero. Then, $y = (1+A)\cdot\alpha$, yet also $y = -(1-A)\cdot\beta$. Thus β is also a simple root within the set $\{\alpha'_l\}$. In other words, the simple root β is located each of the sets $\{\alpha_l\}$ and $\{\alpha'_l\}$. Bearing in mind that for each simple root there is always another simple root that is not orthogonal to it, and continuing the previous argument for each simple root in either set, it is inevitable the two sets of simple roots must coincide. In addition, it emerges that the possible relations amongst the elements of the sets $\{y_k\}$ and $\{y'_k\}$—according to the definition (2.23) and (2.24)—are those previously demonstrated for the a_r Lie algebra case (see, for instance, (2.37) and (2.40)).

3. The Lax pair construction

In the bulk, a Lax pair representation for a theory with r field of which $(r-1)$ represents a conformal a_{r-1} Toda theory and the remaining field a free massless one may have the following form:

$$a_t = \frac{1}{2}\left[\partial_x v\cdot\mathbf{H} + \sum_{i=1}^{r-1}\left(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i}\right)\,\mathrm{e}^{\alpha_i\cdot v/2}\right] + \lambda E_{\alpha_r}\,\mathrm{e}^{\alpha_r\cdot v/2},$$
$$a_x = \frac{1}{2}\left[\partial_t v\cdot\mathbf{H} + \sum_{i=1}^{r-1}\left(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i}\right)\,\mathrm{e}^{\alpha_i\cdot v/2}\right] + \lambda E_{\alpha_r}\,\mathrm{e}^{\alpha_r\cdot v/2}. \tag{3.1}$$

Matrices $\mathbf{H}$ are the generators of the Cartan subalgebra of an a_r Lie algebra whose simple roots are $\alpha_i, i = 1, \ldots, r$ and $E_{\pm\alpha_i}$ are the generators of the simple roots or their negatives. Finally, λ is the spectral parameter. Using the Lie algebra commuting relations

$$[\mathbf{H}, E_{\pm\alpha_i}] = \pm\alpha_i E_{\pm\alpha_i}, \qquad [E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij}\mathbf{H}, \tag{3.2}$$

it can be checked that the Lax pair (3.1) ensures the zero curvature condition

$$\partial_t a_x - \partial_x a_t + [a_t, a_x] = 0, \tag{3.3}$$

is equivalent to the equation of motion. In the present case,

$$\partial^2 v = -\sum_{i=1}^{r-1}\alpha_i\,\mathrm{e}^{\alpha_i\cdot v}. \tag{3.4}$$

Thus, the r components of the vector v, or r linear combinations of these components—depending on the base chosen for the simple roots—represent an a_{r-1} Toda field theory together with a free massless field.

Consider a defect at $x = 0$, which links an a_r Toda field theory on the left with an a_{r-1} Toda field theory and a free massless field on the right. The Lax pair describing such a system may be constructed as explained in [7]. Consider two overlapping regions $R^<$ $(x < b, b > 0)$ and $R^>$ $(x > a, a < 0)$ each containing the defect, and in each region define a new Lax pair as follows:

$$\begin{aligned} R^<:\quad & \hat{a}_t^< = a_t^<(u) - \tfrac{1}{2}\theta(x-a)(u_x - Au_t - Bv_t\psi + \nabla_u\mathcal{D})\cdot\mathbf{H},\\ & \hat{a}_x^< = \theta(a-x)a_x^<(u),\\ R^>:\quad & \hat{a}_t^> = a_t^>(v) - \tfrac{1}{2}\theta(b-x)(v_x - B^T u_t + Av_t - \nabla_v\mathcal{D})\cdot\mathbf{H},\\ & \hat{a}_x^> = \theta(x-b)a_x^>(v), \end{aligned} \tag{3.5}$$

where $a_t^<$ and $a_x^<$ are the Lax pair for an a_r Toda model (see [7]), while $a_t^>$ and $a_x^>$ coincide with the Lax pair (3.1). Applying the zero curvature condition (3.3), the Lax pair (3.5) yields both the equations of motion for the fields u and v in the two regions $x < a$ and $x > b$ and the defect conditions at $x = 0$ and $x = b$. In the overlapping region $a < x < b$ it implies that the fields u and v are independent of x throughout the overlap. On the other hand, maintaining the zero curvature condition within the overlap also requires the two components $\hat{a}_t^<$ and $\hat{a}_t^>$ to be related by a gauge transformation:

$$\mathcal{K}_t = \mathcal{K}\hat{a}_t^> - \hat{a}_t^<\mathcal{K}. \tag{3.6}$$

Setting

$$\mathcal{K} = \mathrm{e}^{-\mathbf{H}\cdot(Au+Bv)/2}\tilde{\mathcal{K}}\,\mathrm{e}^{\mathbf{H}\cdot(B^T u - Av)/2}, \tag{3.7}$$

such that the matrix $\tilde{\mathcal{K}}$ is independent of t, equation (3.6) leads to

$$\tilde{\mathcal{K}}\mathbf{H}\cdot\nabla_v\mathcal{D} + \mathbf{H}\tilde{\mathcal{K}}\cdot\nabla_u\mathcal{D} = \lambda\sum_{i=1}^{r}\mathrm{e}^{\alpha_i\cdot(B^T u+Bv)/2}\big[E_{\alpha_i}, \tilde{\mathcal{K}}\big]$$
$$-\frac{1}{\lambda}\sum_{i=1}^{r}E_{-\alpha_i}\tilde{\mathcal{K}}\,\mathrm{e}^{\alpha_i\cdot B(u-v)/2} + \frac{1}{\lambda}\sum_{i=1}^{r-1}\tilde{\mathcal{K}}E_{-\alpha_i}\,\mathrm{e}^{-\alpha_i\cdot B^T(u-v)/2}.$$

Choosing, as an example, the matrix (2.38), the relation $(\alpha_i B^T) = -(\alpha_{i+1} B)$ holds, and this allows the above expression to be rewritten as follows:

$$\tilde{\mathcal{K}}\mathbf{H}\cdot\nabla_v\mathcal{D} + \mathbf{H}\tilde{\mathcal{K}}\cdot\nabla_u\mathcal{D} = \lambda\sum_{i=1}^{r} \mathrm{e}^{\alpha_i\cdot(B^T u+Bv)/2}\big[E_{\alpha_i}, \tilde{\mathcal{K}}\big] - \frac{1}{\lambda}E_{-\alpha_1}\tilde{\mathcal{K}}\,\mathrm{e}^{\alpha_1\cdot B(u-v)/2} - \frac{1}{\lambda}\sum_{i=2}^{r}\big(E_{-\alpha_i}\tilde{\mathcal{K}} - \tilde{\mathcal{K}}E_{-\alpha_{i-1}}\big)\,\mathrm{e}^{\alpha_i\cdot B(u-v)/2}. \tag{3.8}$$

Bearing in mind the form of the defect potential (2.44), and assuming the following perturbation solution for $\tilde{\mathcal{K}}$:

$$\tilde{\mathcal{K}} = I + \sum_{i=1}^{\infty}\frac{k_i}{\lambda^i}, \tag{3.9}$$

the terms on either side of expression (3.8) must match at each order in λ. It is straightforward to see that this happens for terms of order λ and λ^0 provided $k_1 = \sigma\sum_{i=1}^{r} E_{-\alpha_i}$. However, the terms at order $1/\lambda$ are trickier to analyse. They give

$$\sigma\sum_{i=1}^{r}\big(E_{-\alpha_i}\mathbf{H}\cdot\nabla_v\mathcal{D} + \mathbf{H}E_{-\alpha_i}\cdot\nabla_u\mathcal{D}\big) = \sum_{i=1}^{r}\mathrm{e}^{\alpha_i\cdot(B^T u+Bv)/2}\big[E_{\alpha_i}, k_2\big] - E_{-\alpha_1}\,\mathrm{e}^{\alpha_1\cdot B(u-v)/2} - \sum_{i=2}^{r}\big(E_{-\alpha_i} - E_{-\alpha_{i-1}}\big)\,\mathrm{e}^{\alpha_i\cdot B(u-v)/2}. \tag{3.10}$$

Making use of the defect potential once more (2.44), and of the explicit expression (2.38) for the matrix B, it is possible to compare separately the terms in $1/\lambda$ proportional to $\exp(\alpha_i\cdot B(u-v)/2)$ and $\exp(\alpha_i\cdot(B^T u + Bv)/2)$. The former lead to

$$-\sum_{ij=1}^{r}\frac{1}{2}(\alpha_j\cdot B\alpha_i)E_{-\alpha_i}\,\mathrm{e}^{\alpha_j\cdot B(u-v)/2} = -\sum_{i=2}^{r}\big(E_{-\alpha_i} - E_{-\alpha_{i-1}}\big)\,\mathrm{e}^{\alpha_i\cdot B(u-v)/2} - E_{-\alpha_1}\,\mathrm{e}^{\alpha_1\cdot B(u-v)/2}, \tag{3.11}$$

where the Lie algebra commutation relations (3.2) have been used. Expression (3.11) is clearly an identity. The remaining terms of (3.10), which are proportional to $\exp(\alpha_i\cdot B(u-v)/2)$, lead to an expression for k_2[3]. On the other hand, $k_2 \equiv 0$ when evaluated in an $(r+1)$-dimensional representation for which

$$\big(E_{\alpha_i}\big)_{ab} = \delta_{ai}\delta_{bi-1}, \qquad a, b = 1, \ldots, (r+1). \tag{3.12}$$

Therefore, in this particular representation a complete expression for the element $\tilde{\mathcal{K}}$ is

$$\tilde{\mathcal{K}} = I + \frac{\sigma}{\lambda}\sum_{i=1}^{\infty}E_{-\alpha_i}. \tag{3.13}$$

The existence of a Lax pair representation strongly suggests that the system described in this section is integrable. Note, it is worth emphasizing that a carefully chosen collection of defects arranged along the x-axis is able to link an a_r conformal Toda field theory with r free massless fields. Or, since the a_1 Toda field model is so related to a massless field, the a_r Toda model can be decomposed into a collection of r Liouville models instead, or indeed to a mixture of p Liouville models and q massless free fields with $(p+q) = r$.

Note, in the discussion above to solve expression (3.8) it was supposed that $\tilde{\mathcal{K}}$ was an expansion in inverse powers of the spectral parameter λ. In [7] it was pointed out that $\tilde{\mathcal{K}}$ could

[3] See [7] for details of a similar calculation.

also be regarded as having an expansion in positive powers of λ. In those circumstances a slightly different—yet still consistent—relationship between the matrices A, B and the form of the defect potential $\mathcal{D}$ was found. In the present case, by looking at (3.8), it should be noted that such a possibility is not allowed. In fact, to be able to obtain an alternative solution it would be necessary to start with a different expression to (3.1) for the Lax pair describing an a_{r-1} Toda theory together with a free massless field. The Lax pair representation (3.1) may be replaced by

$$\begin{aligned} a_t &= \frac{1}{2}\left[\partial_x v\cdot\mathbf{H} + \sum_{i=1}^{r-1}\left(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i}\right)\mathrm{e}^{\alpha_i\cdot v/2}\right] - \frac{1}{\lambda}E_{-\alpha_r}\,\mathrm{e}^{\alpha_r\cdot v/2}, \\ a_x &= \frac{1}{2}\left[\partial_t v\cdot\mathbf{H} + \sum_{i=1}^{r-1}\left(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i}\right)\mathrm{e}^{\alpha_i\cdot v/2}\right] + \frac{1}{\lambda}E_{-\alpha_r}\,\mathrm{e}^{\alpha_r\cdot v/2}, \end{aligned} \tag{3.14}$$

which leads—via the zero curvature condition—to the same equations of motion (3.4). Proceeding in a similar manner as before, and using the same matrix B (2.38), the analogue of expression (3.8) is

$$\begin{aligned} \tilde{\mathcal{K}}\mathbf{H}\cdot\nabla_v\mathcal{D} + \mathbf{H}\tilde{\mathcal{K}}\cdot\nabla_u\mathcal{D} &= \lambda\sum_{i=2}^{r}\left(E_{\alpha_i}\tilde{\mathcal{K}} - \tilde{\mathcal{K}}E_{\alpha_{i-1}}\right)\mathrm{e}^{-\alpha_i\cdot B(u-v)/2} + \lambda E_{\alpha_1}\tilde{\mathcal{K}}\,\mathrm{e}^{-\alpha_1\cdot B(u-v)/2} \\ &\quad - \frac{1}{\lambda}\sum_{i=1}^{r}\mathrm{e}^{-\alpha_i\cdot(B^T u+Bv)/2}\left[E_{-\alpha_i},\tilde{\mathcal{K}}\right], \end{aligned} \tag{3.15}$$

which can be solved using an expansion in positive powers of λ for the element $\tilde{\mathcal{K}}$. It should be mentioned that to achieve this a slightly different relationship between the matrices A and B has been used, namely

$$B = -(1+A), \qquad B^T = (-1+A) = (-2-B). \tag{3.16}$$

Expression (3.16) can be obtained by the total momentum conservation analysis of section 2 by looking at the first expression in (2.7). In fact, it can be rewritten in an alternative way with respect to (2.10) as

$$(-1+A^T)(-1+A) = BB^T, \tag{3.17}$$

from which (3.16) follows.

4. Classical a_r affine Toda models with a defect

In this section attention will be focussed on the affine Toda model related to the Lie algebra a_r. To summarize briefly, the model is described by the following Lagrangian density:

$$\mathcal{L}_D = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v - \delta(x)\left(\tfrac{1}{2}u_t\cdot Au + \tfrac{1}{2}v_t\cdot Av + u_t\cdot Bv + \mathcal{D}(u,v)\right). \tag{4.1}$$

The bulk Lagrangian densities $\mathcal{L}_u$ and $\mathcal{L}_v$ are given by (2.1) with all integers n_j equal to one, and $(\alpha_j\cdot\alpha_j) = 2$. The matrix $B = (1-A)$, which is given by the formula (2.38), and satisfies the following:

$$\alpha_k\cdot B\alpha_j = \begin{cases} 2 & k = j, \\ -2 & k = j+1, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 0,\ldots,r, \qquad \alpha_{r+1} = \alpha_0. \tag{4.2}$$

Finally, the defect potential $\mathcal{D}$ is given in (2.41) where σ is the defect parameter. Setting $r = 1$ the Lagrangian (4.1) describes the sinh-Gordon model with a purely transmitting defect, first investigated from this point of view in [6].

The a_r affine Toda model with fields and coupling constant β restricted to be real describes, after quantization, r interacting scalars, also known as fundamental Toda particles, whose classical mass parameters are given by

$$m_a = 2m \sin\left(\frac{\pi a}{h}\right), \qquad a = 1, 2 \ldots, r, \tag{4.3}$$

where $h = (r + 1)$ is the Coxeter number of the algebra. On the other hand, if the fields are permitted to be complex the model possesses classical 'soliton' solutions [16]. Conventionally, in the description of the complex affine Toda field theory the coupling constant β is replaced with $\mathrm{i}\beta$. It is then easy in (4.1) to switch from the real affine Toda model for the Lie algebra a_r to the complex one. In the bulk soliton solutions interpolate between constant zero energy field configurations as x runs from $-\infty$ to ∞. The constant solutions are given by $v = 2\pi\lambda/\beta$, where λ belongs to the weight lattice of the Lie algebra a_r. Each of them is characterized by a topological charge, which is defined as follows:

$$Q = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, u_x = \frac{\beta}{2\pi}\left[\phi(\infty, t) - \phi(-\infty, t)\right], \tag{4.4}$$

and lies in the weight lattice of the algebra. Explicitly, solutions of this type have the form

$$u_a = -\frac{1}{\mathrm{i}\beta}\sum_{j=0}^{r} \alpha_j \ln\left(1 + E_a \omega_a^j\right), \qquad E_a = \mathrm{e}^{a_a x - b_a t + \xi_a}, \qquad \omega_a = \mathrm{e}^{2\pi \mathrm{i} a/h}, \qquad a = 1, \ldots, r \tag{4.5}$$

where $(a_a, b_a) = m_a(\cosh\theta, \sinh\theta)$, θ is the soliton rapidity and ξ_a is a complex parameter, which, though almost arbitrary, must be chosen so that there are no singularities in the solutions as the real coordinates x and t vary. Despite the solutions (4.5) being complex, Hollowood [16] showed that their total energy and momentum are actually real and their masses, at rest, are given by

$$M_a = \frac{2hm_a}{\beta^2}, \qquad a = 1, 2, \ldots, r, \tag{4.6}$$

where m_a are the mass parameters of the real scalar theory (4.3).

For each $a = 1, \ldots, r$ there are several solitons whose topological charges lie in the set of weights of the fundamental ath representation of a_r [17]. However, apart from the two extreme cases, $a = 1$ and $a = r$, not every weight belonging to one of the other representations corresponds to the topological charge of a stationary soliton. The number $\tilde{n}_a$ of possible charges for the representation with label a is exactly equal to h divided by the greatest common divisor of a and h. By shifting the parameter ξ_a by $2\pi a/h$ the soliton solution (4.5) changes its topological charge, since such a shift operates a cycle permutation of the roots $(\alpha_j \to \alpha_{(j-1)})$. Such a permutation is equivalent to the application of the Coxeter element

$$t(\alpha_j) = s_r s_{r-1} \ldots s_2 s_1(\alpha_j), \qquad s_i(\alpha_j) = \alpha_j - (\alpha_j \cdot \alpha_i)\alpha_i. \tag{4.7}$$

Therefore, the relevant weights are orbits of the Coxeter element.

When a defect is introduced, some of the properties previously described will change. For instance, constant field configurations, which are solutions of both the equations of motion and the defect conditions that follow from the Lagrangian (2.1), are given by $(u, v) = (2\pi\lambda_a/\beta, 2\pi\lambda_b/\beta)$, where the label a and b refer to the specific fundamental representations to which the weights λ_a and λ_b belong (up to translations by roots, since energy and momentum are invariant under translations of the fields by elements of the root lattice). Their energy and momentum are now different from zero and equal to (and for convenience, $\sigma = \mathrm{e}^{-\eta}$)

$$(\mathcal{E}_{a,b}, \mathcal{P}_{a,b}) = -\frac{2hm}{\beta^2}\left[\cosh\left(\eta - \frac{2(a-b)\pi\mathrm{i}}{h}\right), -\sinh\left(\eta - \frac{2(a-b)\pi\mathrm{i}}{h}\right)\right], \quad a, b = 1, \ldots, r. \tag{4.8}$$

Note that when the two weights describing the static configurations of the fields u and v belong to the same representation the energy and momentum will be real[4]. Note also that the topological charges carried by a defect constitute a much larger set than the number of possibilities for stationary solitons themselves.

Another interesting change introduced by the defect is represented by the behaviour of a soliton solution which travels through a defect. By convention, a soliton (4.5) with positive rapidity will travel from the left to the right along the x-axis and at some time it will meet the defect located at $x = 0$. The soliton v emerging on the right will be similar to u, but delayed. It is described by

$$v_a = -\frac{1}{\mathrm{i}\beta}\sum_{j=0}^{r} \alpha_j \ln\left(1 + z_a E_a \omega_a^j\right), \tag{4.9}$$

where z_a represents the delay of the soliton travelling through the defect and which, by making use of the defect conditions, is found to be

$$z_a = \left(\frac{\mathrm{e}^{-(\theta-\eta)} - \mathrm{i}\mathrm{e}^{\mathrm{i}\pi a/h}}{\mathrm{e}^{-(\theta-\eta)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi a/h}}\right), \qquad \sigma = \mathrm{e}^{-\eta}. \tag{4.10}$$

This delay is generally complex with exceptions being self-conjugate solitons, corresponding to $a = h/2$ (with r odd), for which the delay is real. Expression (4.10) has a complex simple pole at $\theta = \eta + \mathrm{i}\left(\frac{\pi a}{h} - \frac{\pi}{2}\right)$. This means that a soliton with real rapidity can be absorbed by a defect only if it lies in the self-conjugate representation. This fact was first noted in [6] in the context of the sine-Gordon model. In [10], by examining the argument of the phase of the delay (4.10), it was noted that the defect might induce a phase shift in the soliton that effects a change in the topological charge of the soliton itself, at least provided the shift lies in a suitable range. It was found that the phase shift can be at most equal to $2\pi a/h$ for $a = 1, \ldots, (h-1)/2$ (r even), or $a - 1, \ldots, r/2 - 1$ (r odd). While it is $-2\pi a/h$ for the corresponding anti-solitons $(h - a)$. This quantity should be compared with the quantity separating two different topological charge sectors, which is $2\pi/\tilde{n}_a$. This suggests that a soliton in the first representation or an anti-soliton in the corresponding last representation might convert, at most, to one of the adjacent solitons/anti-solitons within its multiplet as it passes the defect. However, the scope for jumping to configuration other than adjacent soliton increases as the representation investigated moves towards representations associated with more central spots of the Dynkin diagram.

5. A transmission matrix for the a_r affine Toda field theories

In [10] the transmission matrix for the a_2 affine Toda model was thoroughly investigated. A complete classification of the infinite-dimensional solutions of the triangular equation—subject only to a few reasonable assumptions—were obtained. Among them, it was possible to select solutions relevant for the defect problem, and to complete them with a suitably chosen (though not unique) overall scalar factor fixing their zero-pole structure in a minimal

[4] The parameter η is chosen to be real.

way. In this section, the aim is to extend those results to the whole a_r affine Toda series. In [10] the different behaviour of solitons and anti-solitons travelling through a defect was noted. In particular, it was always possible to find a solution for which one group (the $a = 1$ solitons, for example) seemed to match the strict selection rule mentioned above at the end of section 4, which concerned the restricted possibilities for a soliton to change its topological charge, while the other group ($a = 2$) did not. In a sense this was surprising albeit entirely consistent with the requirements of the bootstrap. On the other hand, some differences between solitons and anti-solitons should be expected because of the lack of parity or time-reversal invariance of the Lagrangian describing the defect conditions. It will be seen that this different behaviour between solitons and anti-solitons is found in all a_r affine Toda models, at least for solitons and anti-solitons in the first ($a = 1$) and last ($a = r$) representations, respectively.

The starting point is the set of 'triangle relations' that relate the elements of the transmission matrix T to the elements of the bulk scattering matrix S [1]. They are

$$S_{kl}^{mn}(\Theta)T_{n\alpha}^{t\beta}(\theta_1)T_{m\beta}^{s\gamma}(\theta_2) = T_{l\alpha}^{n\beta}(\theta_2)T_{k\beta}^{m\gamma}(\theta_1)S_{mn}^{st}(\Theta), \tag{5.1}$$

where $\Theta = (\theta_1 - \theta_2)$. Note the presence of two types of labels in the transmission matrix elements. The Roman labels are a finite set of positive integers $1, 2, \ldots, d$ labelling the soliton states within a representation of dimension d, while the Greek labels represent vectors in the weight lattice of the Lie algebra a_r (it is expected that a stable, basic defect will be labelled by the root lattice).

The S-matrices describing the scattering of solitons in the a_r affine Toda field theory were conjectured some time ago by Hollowood [18]. Hollowood's proposal makes use of Jimbo's R-matrices [19], which are trigonometric solutions of the Yang–Baxter equation (YBE) associated with the quantum group $U_q(a_r)$. According to the proposal, the solitons of the model lie in (and fill up) the r different multiplets corresponding for generic q to the r fundamental representations of the algebra $U_q(a_r)$. The number of states in each multiplet coincides with the number of weights in the corresponding representation. For example, the S-matrix $S^{ab}(\Theta)$ describes the scattering of two solitons with rapidities θ_1 and θ_2, lying in the multiplets a and b, respectively. Hence, it is an interwining map on the two representation spaces V_a and V_b

$$S^{ab}(\Theta) : V_a \otimes V_b \longrightarrow V_b \otimes V_a, \qquad S^{ab}(\Theta) = \rho^{ab}(\Theta)R^{ab}(\Theta) \tag{5.2}$$

where R^{ab} is Jimbo's R-matrix and ρ^{ab} is a scalar function determined by the requirements of 'unitarity', crossing symmetry, analyticity and other consistency requirements (such as bootstrap relations), which a scattering matrix ought to satisfy [18].

However, in practice it is enough to know explicitly the S^{11}-matrix—also known as the fundamental scattering matrix—describing the scattering of the solitons in the first representation, since all the other scattering matrices can be obtained from it on applying a bootstrap procedure. The representation space V_1 of the first multiplet has dimension h and its states are labelled by weights representation, which can be written conveniently as follows:

$$l_j^1 \equiv l_j = \sum_{l=1}^{r} \frac{(h-l)}{h}\alpha_l - \sum_{l=1}^{j-1}\alpha_l, \qquad j = 1, \ldots, h. \tag{5.3}$$

Abbreviating $S^{11} \equiv S$, the non-zero elements of S are given by

$$\begin{aligned}
S_{jj}^{jj}(\Theta) &= \rho(\Theta)(qX - q^{-1}X^{-1}),\\
S_{jk}^{kj}(\Theta) &= \rho(\Theta)(X - X^{-1}), \qquad k \neq j,\\
S_{jk}^{jk}(\Theta) &= \rho(\Theta)(q - q^{-1})\begin{cases} X^{(1-2|l|/h)}\ |_{l=j-k<0} \\ X^{-(1-2|l|/h)}\ |_{l=j-k>0} \end{cases}
\end{aligned} \tag{5.4}$$

with

$$X = \frac{x_1}{x_2}, \qquad x_j = \mathrm{e}^{h\gamma\theta_j/2}, \qquad j = 1, 2 \qquad q = -\mathrm{e}^{-\mathrm{i}\pi\gamma}, \qquad \gamma = \frac{4\pi}{\beta^2} - 1. \tag{5.5}$$

(Note, it is assumed $0 < \beta^2 < 4\pi$ so that $0 < \gamma < \infty$.)

The scalar function ρ is given by the following expression:

$$\rho(\Theta) = \frac{\Gamma(1 + h\gamma\mathrm{i}\Theta/2\pi)\Gamma(1 - h\gamma\mathrm{i}\Theta/2\pi - \gamma)}{2\pi\mathrm{i}} \frac{\sinh(\Theta/2 + \mathrm{i}\pi/h)}{\sinh(\Theta/2 - \mathrm{i}\pi/h)} \times \prod_{k=1}^{\infty} \frac{F_k(\Theta)F_k(2\pi\mathrm{i}/h - \Theta)}{F_k(2\pi\mathrm{i}/h + \Theta)F_k(2\pi\mathrm{i} - \Theta)}, \tag{5.6}$$

where

$$F_k(\Theta) = \frac{\Gamma(1 + h\gamma\mathrm{i}\Theta/2\pi + hk\gamma)}{\Gamma(h\gamma\mathrm{i}\Theta/2\pi + (hk+1)\gamma)}.$$

Equipped with the S-matrix, it is possible, in principle, to solve the triangle equations (5.2) to obtain an expression for the transmission matrix T^1 (which will be denoted by T) for solitons lying in the first representation. As a consequence of topological charge conservation, the elements of the transmission matrix will have the following form:

$$T_{i\alpha}^{n\beta}(\theta) = t_{i\alpha}^{n}(\theta)\delta_{\alpha}^{\beta - l_i + l_n} \qquad i, n = 1, \ldots, h, \tag{5.7}$$

where l_i, l_n are the weights (5.3) and α and β lie in the root lattice. However, when this expression is inserted into equation (5.2), one rapidly discovers there are many different solutions. Amongst these there will be the transmission matrix that describes the scattering of a soliton by a defect, itself characterized classically by the choice of the matrix B (2.38). Making use of both the experience acquired in this kind of calculation and the results already obtained for the a_2 affine Toda model, it reasonable to claim that the non-zero elements of the appropriate solution—up to an undetermined scalar function $g(\theta)$—are

$$T_{i\alpha}^{i\beta}(\theta) = g(\theta)Q^{\alpha\cdot l_i}\delta_{\alpha}^{\beta}, \qquad T_{i\alpha}^{(i-1)\beta}(\theta) = g(\theta)(t^{1/h}x^{2/h})\delta_{\alpha}^{\beta - l_i + l_{i-1}},$$
$$i = 1, \ldots, h, \qquad (i-1) = 0 \equiv h,$$

where t is a constant parameter and $Q = -\mathrm{e}^{\mathrm{i}\pi\gamma}$ depends on the coupling constant appearing in the classical Lagrangian density. Setting $t = \mathrm{e}^{-h\gamma\Delta}$ and $\hat{x} = \mathrm{e}^{\gamma(\theta-\Delta)}$, the solution above can be rewritten in the following neater form:

$$T_{i\alpha}^{i\beta}(\theta) = g(\theta)Q^{\alpha\cdot l_i}\delta_{\alpha}^{\beta}, \qquad T_{i\alpha}^{(i-1)\beta}(\theta) = g(\theta)\hat{x}\delta_{\alpha}^{\beta - l_i + l_{i-1}}, \qquad i = 1, \ldots, h, \qquad (i-1) = 0 \equiv h. \tag{5.8}$$

It should be pointed out that suitably designed unitary transformations—of the same type as those used in [10]—have been used to reduce the number of free constants appearing in the solution to just the one essential parameter Δ. Note that solution (5.8) provides a good match with the classical situation because of the presence of zeros in expected positions, meaning that a soliton might convert to only one of its adjacent solitons, thereby respecting the classical selection rules mentioned earlier. Further calculations using the bootstrap—which will not appear in this paper—suggest that this agreement between the classical and the quantum situation with regard to selection rules holds also for the other 'soliton' representations a with $a = 2, 3, \ldots, (h-1)/2$ (r even) or $a = 2, 3, \ldots, (h/2 - 1)$ (r odd), but not for the rest, regarded as 'anti-soliton' representations.

The transmission matrices describing the interaction between a defect and the solitons lying in any of the other representations of the algebra a_r could be computed by applying a bootstrap procedure, which will be described in the following section. Such a procedure together with an additional constraint is also used to obtain the overall factor $g(\theta)$. The argument goes as follows. First of all, the extra constraint is provided by the crossing relation that the solution (5.8) must satisfy, given by

$$T^{(h-a)}{}^{i\beta}_{n\alpha}(\theta) = \tilde{T}^{a}{}^{n\beta}_{i\alpha}(\mathrm{i}\pi - \theta) \qquad a = 1, \ldots, r \tag{5.9}$$

where the matrix $\tilde{T}^a$ describes the interaction between the defect and a soliton within the a representation travelling from the right to the left. In fact, since parity is violated explicitly in the description of the defect, the matrix $\tilde{T}^a$ is expected to differ from the matrix T^a describing solitons travelling from the left to the right. Obviously, the matrix $\tilde{T}^a$ itself satisfies a set of triangular equations albeit with a different, though related, S-matrix. These equations differ in some details from (5.1), and therefore $\tilde{T}^a$ is not amongst its solutions. Note, however, that matrices T^a and $\tilde{T}^a$ must be related to each other, and it is natural to suppose the following:

$$T^{a}{}^{b\beta}_{a\alpha}(\theta)\tilde{T}^{a}{}^{c\gamma}_{b\beta}(-\theta) = \delta^c_a\delta^\gamma_\alpha. \tag{5.10}$$

Note that for the sine-Gordon model, which is the only affine Toda field theory in the a_r series to be unitary, expression (5.10) is equivalent to the unitarity condition since in that case $\tilde{T}(-\theta) \equiv (T(\theta))^\dagger$. Thus, by computing the inverse of solution (5.8), it is possible to obtain the transmission matrix—again up to a multiplicative factor—for the solitons within the representation r, which are in fact anti-solitons with respect to the solitons in the first representation. The elements of this matrix read

$$T^{r}{}^{(i+j)\beta}_{i\alpha}(\theta) = \frac{\hat{x}^j\, Q^{-\alpha\cdot k_{(i+j)}}}{g(\theta - \mathrm{i}\pi)(1-\hat{x}^h Q^{-1})}\delta_\alpha^{\beta+l_i-l_{i+j}}, \tag{5.11}$$

with

$$i = 1, \ldots, h \qquad j = 0, \ldots, (h-1) \qquad k_{(i+j)} = l_i + l_{i+1} + \ldots + l_{i+j},$$

where it must be borne in mind that $(i+j)$ is evaluated mod(h). It should be remarked that the weights in the representation r are $-l_i$, with l_i given by (5.3). Note that this time, the solution (5.11) does not possess the expected zeros corresponding to the classical selection rule. Each anti-soliton may convert into any of the anti-soliton within the same representation, though the classically allowed transmission remains the most probable.

Comparing the solution (5.11), obtained by applying the crossing relation, with the solution for the same anti-solitons that will be computed in the following section, it is possible to constrain the overall function $g(\theta)$ and find an explicit expression for it.

6. Bootstrap procedure and the overall factor of the transmission matrix

Consider D_α to be a formal operator representating the defect. Then, it is natural to describe the interaction between a defect and a soliton within the representation a as follows:

$$A^a_i(\theta)D_\alpha = T^{a}{}^{j\beta}_{i\alpha}(\theta)D_\beta A^a_j(\theta), \tag{6.1}$$

where A^a_i is set of operators representing the soliton state i in the representation a. The total number of states in the representation a is $h!/(a!(h-a)!)$, and, in principle, by making use of the h states within the first representation, all other states can be built. Hence, expression (6.1) allows us to construct all transmission matrices simply relying on the $T^1 \equiv T$-matrix (5.8). The construction of the soliton states can be elucidated using an iterative process. Consider the states l^2_k in the second representation. Since each weight is $l^2_i = l_j + l_k$ where l_j, l_k are the weights (5.3) with $j \neq k$. The corresponding state is given, schematically, by

$$A^2_{i\{jk\}}(\theta) \equiv {}^{11}c_i{}^{jk} A_j(\theta - \mathrm{i}\pi/h)A_k(\theta + \mathrm{i}\pi/h) + {}^{11}c_i{}^{kj} A_k(\theta - \mathrm{i}\pi/h)A_j(\theta + \mathrm{i}\pi/h), \tag{6.2}$$

where $\Theta = \mathrm{i}2\pi/h$ is the location of the simple pole in the scattering matrix $S^{11} \equiv S$ corresponding to a soliton in the second representation. The constants ${}^{11}c_i{}^{jk}$ and ${}^{11}c_i{}^{kj}$ are the couplings, whose ratio—effectively the only data needed—can be calculated using the scattering matrix S.

The next representation is the third one, and to construct its states more care must be taken. The argument goes as follows. The weights associated with each state can be written as $l_i^3 = l_j + l_k + l_m$ with $j \neq k \neq m$, that is making use only of the weight in the first representation. Formally, such a state can be formulated as follows:

$$A^3_{i\{jkm\}}(\theta) \equiv {}^{12}c_i{}^{jp} A_j(\theta - \mathrm{i}2\pi/h) A^2_{p\{km\}}(\theta + \mathrm{i}\pi/h) + {}^{12}c_i{}^{kq} A_k(\theta - \mathrm{i}2\pi/h) A^2_{q\{jm\}}(\theta + i\pi/h) + {}^{12}c_i{}^{mt} A_m(\theta - \mathrm{i}2\pi/h) A^2_{t\{jk\}}(\theta + \mathrm{i}\pi/h), \tag{6.3}$$

where the coupling ratios can be calculated using the scattering matrix S^{12}. The pole in this matrix corresponding to a soliton in the third representation is located at $\Theta = \mathrm{i}3\pi/h$. Note that an equivalent formulation could have been provided by using the matrix S^{21}. The relevant pole is still located at $\Theta = \mathrm{i}3\pi/h$, but the expression for a soliton state in the third representation would have been

$$A^3_{i\{jkm\}}(\theta) \equiv {}^{21}c_i{}^{pj} A^2_{p\{km\}}(\theta - \mathrm{i}\pi/h) A_j(\theta + \mathrm{i}2\pi/h) + {}^{21}c_i{}^{qk} A^2_{q\{jm\}}(\theta - \mathrm{i}\pi/h) A_k(\theta + i2\pi/h) + {}^{21}c_i{}^{tm} A^2_{t\{jk\}}(\theta - \mathrm{i}\pi/h) A_m(\theta + \mathrm{i}2\pi/h). \tag{6.4}$$

In fact, given three soliton states $A_j(\theta_1)A_k(\theta_2)A_m(\theta_3)$, expression (6.3) describes the case in which first the solitons described by $A_k(\theta_2)$ and $A_m(\theta_3)$ combine together to form a soliton $A_p^2(\theta')$ and subsequently, $A_j(\theta_1)$ and $A_p^2(\theta')$ form the soliton $A_q^3(\theta'')$. On the other hand, expression (6.4) corresponds to a situation where the solitons represented by $A_j(\theta_1)$ and $A_k(\theta_2)$ combine first to give a soliton in the second representation, and so on.

Similarly, and with even more care, it is possible to construct all soliton states on recognizing that the pole corresponding to a soliton in the c representation is located at $\Theta = \mathrm{i}(a+b)\pi/h$ in the scattering matrix S^{ab} with $c = a + b$.

Applying (6.1) to the soliton states in the r representation, it is possible to find

$$T^{r\,(i+j)\beta}_{\ \,i\alpha}(\theta) = f_r(\theta)\hat{x}^j\, Q^{-\alpha\cdot k_{(i+j)}}\, \delta_\alpha^{\beta+l_i-l_{i+j}}, \tag{6.5}$$

where

$$f_r(\theta) = \prod_{a=0}^{r/2-1} g(\theta - \mathrm{i}(2a+1)\pi/h) g(\theta + \mathrm{i}(2a+1)\pi/h), \tag{6.6}$$

if r is even, and

$$f_r(\theta) = g(\theta) \prod_{a=1}^{(r-1)/2} g(\theta - \mathrm{i}2a\pi/h) g(\theta + \mathrm{i}2a\pi/h), \tag{6.7}$$

if r is odd. Note that the latter formula holds for $r \neq 1$, since the bootstrap cannot be applied in the sine-Gordon situation. In that case, one has simply $f_r(\theta) = g(\theta)$. Finally, the solution (6.5) may be compared with the solution (5.11), to provide a constraint for the scalar function $g(\theta)$. It reads

$$g(\theta + \mathrm{i}2\pi) = g(\theta) \frac{(1 + \hat{x}^h(-Q)^r)}{(1 + \hat{x}^h(-Q)^{r+2})}. \tag{6.8}$$

for which a minimal solution—for all a_r affine Toda models—is

$$g(\theta) = \frac{\hat{g}(\theta)\hat{x}^{-1/2}}{2\pi} \tag{6.9}$$

with

$$\hat{g}(\theta) = \Gamma[1/2 + (r/2)\gamma - z] \prod_{k=1}^{\infty} \frac{\Gamma[1/2 + (hk + r/2)\gamma - z]\Gamma[1/2 + (hk - 1 - r/2)\gamma + z]}{\Gamma[1/2 + (hk - r/2)\gamma - z]\Gamma[1/2 + (hk - r/2)\gamma + z]}, \tag{6.10}$$

where

$$\hat{x} = e^{\gamma(\theta - \Delta)}, \qquad z = \frac{ih\gamma(\theta - \Delta)}{2\pi}.$$

The technique adopted in this section can be extended to all representations, and in principal all transmission matrices T^a with $a = 1, \ldots, r$ can be found. For the overall scalar function $g^a(\theta)$ a compact formula reads

$$g^a(\theta) = \frac{\hat{g}^a(\theta)\hat{x}^{-a/2}}{2\pi} \qquad a = 1, \ldots, r \tag{6.11}$$

with

$$\hat{g}^a(\theta) = \Gamma[1/2 + (h - a)\gamma/2 - z] \times \prod_{k=1}^{\infty} \frac{\Gamma[1/2 + (hk + (h - a)/2)\gamma - z]\Gamma[1/2 + (hk - (h + a)/2)\gamma + z]}{\Gamma[1/2 + (hk - (h - a)/2)\gamma - z]\Gamma[1/2 + (hk - (h - a)/2)\gamma + z]}. \tag{6.12}$$

The Γ-function outside the product contains an interesting complex pole, which is located at

$$\theta_a = \Delta - i\vartheta_a, \qquad \vartheta_a = \frac{\pi(h - a)}{h} + \frac{\pi}{h\gamma} \qquad a = 1, \ldots, r. \tag{6.13}$$

Comparing this pole with the pole appearing in the classical delay (4.10), it is possible to relate the defect parameter σ to the complex parameter Δ. Given that in the classical limit $1/\gamma \to 0$, the identification of the two poles (6.13) and (4.10) requires

$$\Delta = \eta + \frac{i\pi}{2}, \qquad \sigma = e^{-\eta}.$$

The complex energy associated with this pole is

$$E_a = M_a \cosh\theta_a = M_a \cosh\eta \sin\vartheta_a + iM_a \sinh\eta \cos\vartheta_a, \tag{6.14}$$

where M_a is the mass of a soliton in the representation a given by (4.6). Provided (6.14) enjoys a positive real part and a negative imaginary part, that is

$$\pi/2 \leqslant \vartheta_a < \pi, \tag{6.15}$$

the pole (6.13) corresponds to an unstable bound state.

Consider first a soliton lying in a representation labelled by $a \leqslant (h/2-1)$ or $a \leqslant (h-1)/2$, depending whether h is even or odd. Then, bearing in mind that $1/\gamma$ is always a positive—or, in the classical limit, zero—quantity, condition (6.15) is satisfied provided $1/\gamma < (h/2 - 1)$ or $1/\gamma < (h - 1)/2$, respectively. Note that in the classical limit ($1/\gamma \to 0$) the energy (6.14) is typically complex and appears to correspond to the energy of an unstable bound state, which could be identified in the classical version of the model as one of the defects whose energy is given by (4.8) (taking $u = 0$, $v = 2\pi\lambda_a/\beta$). In fact, only if h is even and $a = h/2$ does $\vartheta_a \to \pi/2$ in the classical limit, meaning the imaginary part of (6.14) disappears leaving a real part equal to the energy of a soliton a, which moves with rapidity η. This situation corresponds to the classical possibility for a self-conjugate soliton to be infinitely delayed by the defect. Finally, if the soliton lies in a representation, $a \geqslant (h/2 + 1)$ or $a \geqslant (h + 1)/2$, (meaning it is an 'anti-soliton' according to the convention used so far), again, depending whether h is even or odd, an unstable bound state appears within a range $(a - h/2) \leqslant 1/\gamma < a$ of the coupling that does not include a neighbourhood of the classical limit. In other words, these quantum unstable states would be disconnected from any phenomenon occurring in the classical models. That the different representations behave differently in this context appears to compound the difficulties in comparing the quantum theory of these models with the classical theory; not only do real states go missing but unstable states appear unexpectedly. Perhaps these phenomena are related.

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